

# Contingent Claim Pricing using Probability

## Distortion Operators:

Methods from Insurance Risk Pricing and  
their Relationship to Financial Theory\*

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September 20, 2001

**Abstract**

This paper considers the pricing of contingent claims using an approach developed and used in insurance pricing. The approach is of interest and significance because of the increased integration of insurance and financial markets and also because insurance related risks are trading in financial markets as a result of securitisation and new contracts on futures exchanges. This approach uses probability distortion functions as the dual of the utility functions used in financial theory. The pricing formula is the same as the Black-Scholes formula for contingent claims when the underlying asset price is log-normal. The paper compares the probability distortion function approach with that based on financial theory. The theory underlying the approaches is set out and limitations on the use of the insurance based approach are illustrated. We extend the probability distortion approach to the pricing of contingent claims for more general assumptions than those used for Black-Scholes option pricing.

**Keywords:** Contingent claim pricing, Probability distortion functions, Non-expected utility, Insurance pricing, Black and Sholes

**Acknowledgement 1** *The authors gratefully thank Shaun Wang for helpful discussion. Financial support under ARC Large Grant A79803988 is acknowledged as is support from The Institute of Actuaries of Australia.*

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\*This is a work in progress based on part of the Ph.D thesis of the first author.

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# 1 Introduction

With the convergence of financial and insurance markets, there has been a major increase in the trading and securitisation of insurance and similar risks including weather and energy derivatives. Techniques used in insurance pricing are being considered for the pricing of derivative contracts including those for energy risk traded on exchanges such as the CBOT. This has resulted in an interest in insurance pricing and how these techniques relate to financial theory.

At the same time there have been some recent developments in models for pricing insurance risks based on an approach originally developed by Wang (1996 [15]). This approach uses a certainty equivalent and a non-expected utility framework to determine the price to be charged for an insurance risk. The approach can be applied to reinsurance contracts since these can be considered as contingent claims on an underlying insurance risk. More recently the approach has been developed further and applied to financial risks as well as weather risk. Wang (2000 [16]) proposes a form of this insurance risk pricing approach with its foundations in non-expected utility theory that, assuming a normal distribution of returns, is consistent with financial theory. We formally consider this approach and show that it produces the Black-Scholes option pricing formula for contingent claims.

In this paper we consider this approach further and relate it to financial theory. Contingent claims in financial markets are usually valued using no-arbitrage assumptions. If financial markets are arbitrage-free then the current price of a security or contingent claim must be equal to the expected value of its future payoff under a risk neutral measure, discounted to present values at the risk free rate. If the financial market is complete and all contingent claims are attainable using traded securities, then the arbitrage-free price is unique. In incomplete markets, the arbitrage-free price is not necessarily unique and the arbitrage-free approach can only bound the contingent claim price. In these cases an equilibrium approach can be used, in which all investors act so as to maximise the expected utility of consumption and the equilibrium prices

for contingent claims are those for which net supply is zero.

Although the approach developed by Wang, based on a particular form of non-expected utility certainty equivalent, is the same as the Black-Scholes contingent claim price when the underlying security prices are log-normally distributed, a natural question arises as to whether the use of this approach is appropriate in the non-normal case. Wang has recently proposed a modification of the approach that aims to price contingent claims on securities or risks with a non-normal distribution. The method involves a calibration of a market price of risk parameter to an underlying non-normal price distribution using the normal distribution based formula. The calibrated formula is then used to price contingent claims.

This paper formally studies the non-normal case and demonstrates the limitation of using Wang's approach in this case by considering the Cox, Ross and Rubinstein binomial lattice model and the constant elasticity of variance model for security prices.

## **2 Certainty Equivalent, Risk Aversion and Probability Distortion Functions**

In expected utility theory, risk aversion is characterized by a utility function which is concave and increasing in wealth. Individuals are risk averse if they are unwilling to accept a fair gamble that leaves their wealth unchanged. If  $X$  is a risk (random variable) such that  $\mathbb{E}[X] = 0$  (fair gamble), and  $W$  is the individual's wealth, then for a risk-averse individual

$$u(W) > \mathbb{E}[u(W + X)]$$

where  $u$  is the individual's utility function.

To assume such a risk, a risk-averse individual needs to receive a risk premium  $\Pi(X)$  such

that:

$$u(W + \Pi(X)) = \mathbb{E}[u(W + X)]$$

$W + \Pi(X)$  is called the certainty equivalent of  $W + X$ , as it represents the sum of money which, when received with certainty is considered as equivalent to  $W + X$ .

Rearranging gives

$$\Pi(X) = u^{-1}(\mathbb{E}[u(W + X)]) - W$$

and if the utility function is such that

$$u^{-1}(\mathbb{E}[u(W + X)]) = W + u^{-1}(\mathbb{E}[u(X)])$$

then

$$\Pi(X) = u^{-1}(\mathbb{E}[u(X)])$$

will be independent of current wealth.

Yaari (1987 [18]) developed a dual theory of choice under risk that is the dual of expected utility. Wang's approach to pricing insurance risks is consistent with Yaari's dual theory of choice. In the dual theory, the certainty equivalent of a risk  $X$  is defined as:

$$\Pi(X) = \int g(S_X(x))dx = \int xg'(S_X(x))dF_X(x) \tag{1}$$

where  $S_X(x) = P[X > x]$ , and  $g$  is an increasing differentiable function with  $0 < g(u) < 1$  and  $g(0) = 0$  and  $g(1) = 1$ . The function  $g$  is referred to as a probability distortion function. For

expected utility, the utility function is applied to wealth  $x$  and is then weighted by probabilities. In the dual theory the probability distortion function is applied to survival probabilities and is linear in wealth.

Noting that the expected value is given by

$$\mathbb{E}[X] = \int x dF_X(x)$$

and comparing  $\Pi(X)$  to  $\mathbb{E}[X]$ ,  $\Pi(X)$  can be thought of as a corrected mean of  $X$  where the payment  $x$  receives a weight  $g'(S_X(x)) \geq 0$ . Note that these weights sum to 1 since

$$\begin{aligned} \int g'(S_X(x)) dF_X(x) &= \int \frac{d}{dx}[-g(S_X(x))] dx \\ &= g(1) - g(0) \\ &= 1 \end{aligned}$$

If  $g$  is convex, then

$$x_1 > x_2 \Rightarrow S_X(x_1) < S_X(x_2) \Rightarrow g'(S_X(x_1)) < g'(S_X(x_2))$$

Therefore, the weight assigned to a high outcome is less than the weight assigned to a low outcome. Hence, by distorting the tail distribution with a convex function, an individual behaves pessimistically, in the sense that they assign a higher probability to low outcomes and lower probability to high outcomes.

The Arrow-Pratt measure of absolute risk aversion<sup>1</sup> is a local measure of risk-aversion for a utility function, as used in the certainty equivalent assuming expected utility. In a similar fashion to the utility function,  $g$  has associated with it a measure of risk aversion. The comparison of

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<sup>1</sup>Arrow-Pratt's absolute risk aversion is defined as  $A(W) = -\frac{u''(W)}{u'(W)}$

risk aversion in this framework is naturally based on the convexity of the function  $g$  representing the agent's preference function. The more convex the function  $g$ , the more risk averse the agent. The dual Arrow-Pratt risk aversion would be in this case  $\frac{g''(p)}{g'(p)}$  for  $0 < p < 1$ , as defined in Yaari (1986 [18]). In the sense of Ross (1981 [12]), agents are strongly more risk averse, if they require a larger compensation for any mean preserving spread in their prospects, even if the initial situation is not one of perfect certainty. Risk aversion measurement in the sense of Yaari (1986 [18]) and Ross (1981 [12]) is discussed in Röel (1985 [11]).

To sum up, while risk aversion in utility theory is measured by the utility function, in the dual theory, it is measured by the probability distortion function. The choice of the probability distortion function  $g$  determines the properties of the certainty equivalent.

In the literature, Wang (1996 [15]) proposes a general class of probability distortion functions to use in determining insurance premiums. When the probability distortion function is a power function,  $g(x) = x^r$ , the mapping

$$S_X(x) \rightarrow g(S_X(x))$$

is called the Proportional Hazards transform. Although the Proportional Hazards transform has properties that are considered appropriate for insurance pricing, the transform is inconsistent in the treatment of assets and liabilities.

Wang (2000 [16]) proposes another class of distortion functions

$$g_\alpha(u) = \Phi[\Phi^{-1}(u) + \alpha]$$

where  $\Phi(u) = \int_{-\infty}^u \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$  is the standard normal cumulative distribution function.

Van der Hoek and Sherris (2001 [14]) considered the probability distortion function approach and proposed the use of two different distortion functions,  $g$  and  $h$ , to allow a different treatment

of the upside and downside of the risk.

It is important to recognise that certainty equivalents are not market prices for risks. This issue is discussed in Landsman and Sherris (2001, [8]).

### 3 Wang's Probability Distortion Function and Asset Pricing

Wang (2000 [16]) recognized that the Proportional Hazards probability transform can not be applied consistently to assets and liabilities simultaneously. He proposed a different class of probability distortion function that aims to integrate financial and actuarial insurance pricing theories. The probability distortion function proposed is based on the standard cumulative normal distribution. In his paper Wang states that the new distortion function connects four different approaches:

1. the traditional actuarial standard deviation principle,
2. Yaari's economic theory of risk,
3. CAPM, and
4. option-pricing theory.

We will develop the approach in this section of the paper. Consider  $X$  a random variable with a decumulative distribution function  $S_X(x) = P[X > x]$ . Let  $\Phi(x) = \int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$  be the standard normal cumulative distribution function and define

$$g_\alpha(p) = \Phi[\Phi^{-1}(p) + \alpha]$$

for  $p$  in  $[0, 1]$ . This operator shifts the  $p^{th}$  quantile of  $X$ , assuming that  $X$  is normally distributed, by a positive or negative value  $\alpha$  and re-evaluates the normal cumulative probability for the

shifted quantile. Wang shows that  $g_\alpha(p)$  is concave for positive  $\alpha$  and convex for negative  $\alpha$ . In fact it is easy to see that if  $\alpha > 0$ , then  $g_\alpha(p) > p$ , and if  $\alpha < 0$ , then  $g_\alpha(p) < p$ . Since  $g_\alpha$  is continuous and  $g_\alpha(p) \in [0, 1]$ , then it follows that

$g_\alpha$  is convex if  $\alpha < 0$

$g_\alpha$  is concave if  $\alpha > 0$

Under this distortion function, an individual behaves pessimistically by shifting the quantiles to the left, thereby assigning higher probabilities to low outcomes, and behaves optimistically by shifting the quantiles to the right thereby assigning higher probabilities to high outcomes.

Wang (2000 [16]) defines the risk-adjusted premium for a risk  $X$ , by the Choquet integral representation

$$H[X; \alpha] = \int_{-\infty}^0 \{g_\alpha[S_X(x)] - 1\} dx + \int_0^\infty g_\alpha[S_X(x)] dx$$

where  $X$  will be negative if it is an insurance loss and will take positive values for the pay-off from a limited-liability asset. Note that the expectation of  $X$  can be written as follows<sup>2</sup>:

$$\mathbb{E}[X] = \int_{-\infty}^0 [S_X(x) - 1] dx + \int_0^\infty S_X(x) dx$$

When  $X$  is a positive random variable, the Choquet integral representation has the simpler form:

$$H[X; \alpha] = \int_0^\infty g_\alpha[S_X(x)] dx$$

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<sup>2</sup>See the Appendix for details

The tail probability distribution  $S_X(x)$  is distorted by the function

$$g_\alpha(p) = \Phi[\Phi^{-1}(p) + \alpha]$$

Properties of this probability distortion function are included in Wang (2000 [16]). In particular, the cases where the risk has a normal distribution and where the risk has a log-normal distribution are considered.

If a risk  $X$  has a normal distribution,  $N(\mu, \sigma^2)$ , then

$$H[X; \alpha] = \mu + \alpha\sigma$$

and if  $(\ln X) \sim N(\mu, \sigma^2)$ , then

$$H[X; \alpha] = e^{\mu + \alpha\sigma + \frac{\alpha^2}{2}}$$

For completeness these results are derived in the Appendix.

Other properties of interest, that do not depend on the distribution of  $X$ , are:

- For a constant  $c$ ,  $H[c; \alpha] = c$  and  $H[X + c; \alpha] = H[X; \alpha] + c$ ,
- $H[X; \alpha] < E[X]$  if  $\alpha < 0$  and  $H[X; \alpha] > E[X]$  if  $\alpha > 0$ ,
- For  $b > 0$ ,  $H[bX; \alpha] = bH[X; \alpha]$ ,
- For  $b < 0$ ,  $H[bX; \alpha] = bH[X; -\alpha]$  and as a special case  $H[-X; \alpha] = -H[X; -\alpha]$ .

Wang (2000 [16]) develops an approach to asset pricing by applying  $H[X; \alpha]$  to the present value of an asset where the present value is determined by discounting at the risk free interest rate. The economic assumptions underlying the approach amount to treating the certainty equivalent as a market price. The  $H[X; \alpha]$  can be considered as the certainty equivalent of a

risky cash flow. In order to derive pricing results using  $H[X; \alpha]$ , an implied  $\alpha$  is derived that ensures the certainty equivalent is consistent with the market price of the asset. This calibrates the certainty equivalent to the market price.

Wang considers both an horizon of one year and also a multiperiod horizon. For the single period horizon let  $X_i(0)$  be the asset price at time 0 of a security  $i$ , and  $X_i(1)$  the pay-off of the security at time 1. The per period return compounded annually is denoted by  $R_i = \frac{X_i(1)}{X_i(0)} - 1$  and  $r_f$  is the risk-free rate.

Assuming that  $R_i$  is normally distributed with a mean  $\mathbb{E}[R_i]$  and a standard deviation  $\sigma[R_i]$  then

$$H[R_i; -\alpha_i] = \mathbb{E}[R_i] - \alpha_i \sigma[R_i]$$

Wang refers to  $H[R_i; -\alpha_i]$  as the risk-adjusted rate of return for security  $i$ .

Now if  $\alpha_i$  is selected so as to equal

$$\alpha_i = \frac{\mathbb{E}[R_i] - r_f}{\sigma[R_i]}$$

then

$$\begin{aligned} H[R_i; -\alpha_i] &= \mathbb{E}[R_i] - \frac{\mathbb{E}[R_i] - r_f}{\sigma[R_i]} \sigma[R_i] \\ &= r_f \end{aligned}$$

For this selection of  $\alpha$ , using the properties of  $H[X; \alpha]$ , the certainty equivalent will be

$$\begin{aligned}
H\left[\frac{X_i(1)}{1+r_f}; -\alpha_i\right] &= \frac{1}{1+r_f} H[X_i(1); -\alpha_i] \\
&= \frac{X_i(0)}{1+r_f} [H[(1+R_i); -\alpha_i]] \\
&= \frac{X_i(0)}{1+r_f} [1 + H[R_i; -\alpha_i]] \\
&= X_i(0)
\end{aligned}$$

where the result relies on the assumption that the per period return is normally distributed and that  $\alpha_i$  is selected so that  $H[R_i; \alpha]$  equals the risk-free rate.

Wang considers the multi-period horizon case and by a limiting argument derives results for a continuous time model. He assumes that the security price  $X_i(t)$  follows geometric Brownian motion with

$$\frac{dX_i(t)}{X_i(t)} = \mu_i dt + \sigma_i dW_i$$

where  $dW_i$  is an increment in a Brownian motion. In the continuous-time case, the choice of

$$\alpha_i = \frac{\mu_i - r_c}{\sigma_i} \sqrt{T}$$

where  $r_c$  denotes the continuously compounded risk-free rate, results in  $g_{-\alpha_i}$  transforming the asset price distribution  $S_{X_i(T)}$  to  $S_{Y_i(T)}$  where

$$\ln Y_i(T) \sim Normal\left[\ln X_i(0) + \left(r_c - \frac{1}{2}\sigma_i^2\right)T, \sigma_i^2 T\right]$$

Wang’s pricing approach gives

$$X_i(0) = e^{-r_e T} H[X_i(T); -\alpha_i]$$

Wang states that this results follows since the

“no arbitrage condition (or simply market value concept) implies that the risk-adjusted present value of future stock price must equal the current stock price”.

For the case where

$$X_i(T) = (X_T - K)^+$$

the certainty equivalent discounted at the risk free interest rate is the same as the Black-Scholes formula. A formal proof is given later in this paper.

## 4 Financial Theory

Financial theory has been developed for various models of pricing risky assets including contingent claims. The key results are covered in many texts including, for example, Cochrane (2001, [3]), Hunt and Kennedy (2000, [7]) and Pliska (1997, [10]). We will use the notation and approach set out in Cochrane (2001, [3])

The most basic assumption underlying much of the theory underlying contingent claim pricing is that an economy is arbitrage-free. In this case, an economy is arbitrage-free if and only if, in a single period model, the asset price is given by

$$X_i(0) = \mathbb{E}_P [mX_i(1)]$$

where  $m$  is a positive random variable referred to as a stochastic discount factor and the expect-

tation is taken with respect to the “real-world” probability measure,  $P$ .

In general, the law of one price guarantees the existence of a stochastic discount factor  $m$ . No-arbitrage guarantees that the discount factor is strictly positive so that the price of an asset can be written as

$$p = \mathbb{E}[mX]$$

for any asset payoff  $X$  (see Cochrane, Chapter 4, [3]). The CAPM, Black-Scholes pricing and term structure models for bond prices are special cases.

Furthermore, if there is a security with positive price and strictly positive pay-offs in all future states, say security  $k$ , then the economy is arbitrage-free if and only if there exists a probability measure  $Q$ , equivalent to  $P$ , such that

$$\frac{X_i(0)}{X_k(0)} = \mathbb{E}_Q \left[ \frac{X_i(1)}{X_k(1)} \right]$$

and the measure  $Q$  is referred to as an equivalent martingale measure with respect to  $X_k$ . If markets are complete then the equivalent martingale measure  $Q$  will be unique.

Complete markets ensure that any contingent claim will be attainable. This means that the existing assets in the economy can be used to replicate the contingent claim with a self-financing trading strategy.

The asset price for an economy that is arbitrage-free with complete markets is therefore

$$X_i(0) = X_k(0) \mathbb{E}_Q \left[ \frac{X_i(1)}{X_k(1)} \right]$$

If a risk-free asset exists that has initial price of \$1 and pay-off  $(1 + r)$  then

$$X_i(0) = \frac{1}{(1 + r)} \mathbb{E}_Q [X_i(1)]$$

and in this case  $Q$  is the risk-neutral probability measure.

These pricing approaches involve a change of measure to the equivalent martingale measure, or to the risk neutral measure where the risk free security is the numéraire, and the price is derived as an expected value under the new measure.

The multi-period finite horizon economy is a natural extension of the single period case. Markets now need to be dynamically complete in order for contingent claims to be attainable. Dynamic self-financing trading strategies can be used to replicate the contingent claim. In the multi-period case, under an equivalent probability measure  $Q$ , the asset price process in terms of the numéraire asset  $k$  is a martingale. Under the risk-neutral probability measure the asset price discounted at the risk-free rate to present values is a martingale.

Markets, including markets for insurance and energy risk, are often incomplete. In order to price contingent claims in these markets it is necessary to use assumptions about the agents in the economy and their preferences in order to determine an equivalent martingale measure to use for pricing. One approach is based on expected utility.

Consider an economy at time  $t$  consisting of individuals with utility function over consumption given by

$$U(c_t, c_{t+1}) = u(c_t) + \beta \mathbb{E}_t [u(c_{t+1})]$$

where  $c_t$  is consumption at time  $t$ ,  $u(\cdot)$  is an increasing concave utility function,  $\beta$  is a subjective discount factor and  $\mathbb{E}_t$  is the conditional expectation operator over future states at time  $t + 1$ . Individuals maximise this utility function by selecting the optimal amount of each security to

hold, including contingent claims, and their optimal consumption.

Assume that the optimal level of consumption is  $C_t^*$  and  $C_{t+1}^*(s)$  in state  $s$  at time  $t + 1$ . Security  $i$  has pay-off at time  $t + 1$  of  $X_i(t + 1, s)$  in state  $s$  and is assumed to have price  $P_i(t)$  at time  $t$ . Denote by  $\xi > 0$  the additional amount of security  $i$  purchased at price  $P_i(t)$  over and above the optimal holding.

The individual will select  $\xi$  for any given price by solving the following problem

$$J(c_t) = \max_{\{\xi\}} [u(c_t) + \beta \mathbb{E}_t [u(c_{t+1})]]$$

such that

$$c_t = C_t^* - P_i(t) \xi$$

$$c_{t+1}(s) = C_{t+1}^*(s) + X_i(t + 1, s) \xi \text{ for all } s$$

Substituting the constraints into the objective gives

$$J(c_t) = \max_{\{\xi\}} [u(C_t^* - P_i(t) \xi) + \beta \mathbb{E}_t [u(C_{t+1}^* + X_i(t + 1) \xi)]]$$

Differentiating with respect to  $\xi$  and setting  $\xi$  to zero gives the condition for optimal holding of the security  $i$  and optimal consumption at the equilibrium as follows

$$\frac{\partial}{\partial \xi} J(c_t) |_{\xi=0} = -P_i(t) u'(C_t^*) + \beta \mathbb{E}_t [X_i(t + 1) u'(C_{t+1}^*)]$$

so that

$$P_i(t) = \mathbb{E}_t \left[ \beta \frac{u'(C_{t+1}^*)}{u'(C_t^*)} X_i(t + 1) \right]$$

If we let the stochastic discount factor be

$$m = \beta \frac{u'(C_{t+1}^*)}{u'(C_t^*)}$$

then

$$P_i(t) = \mathbb{E}_t[mX_i(t+1)]$$

where expectation is taken with respect to the “real world” probability measure  $P$ .

If a risk free security exists then an investment of 1 in this security will pay-off  $(1+r)$  in one time period so that

$$1 = \mathbb{E}_t[m(1+r)]$$

or

$$\frac{1}{(1+r)} = \mathbb{E}_t[m]$$

Determining prices of contingent claims in an arbitrage-free economy requires the form of the stochastic discount factor to be specified.

## 5 Contingent Claim Pricing

The financial theory used to derive the Black-Scholes option pricing formula usually assumes the underlying security prices follow a geometric Brownian motion. In this case the no-arbitrage approach involving a change of measure is shown to be equivalent to the probability distortion function approach developed in the insurance literature. In other words, pricing a contingent

claim using risk neutral valuation by taking the expectation of the payoff under the  $Q$  measure is the same as the insurance premium determined by Wang's probability distortion function.

## 5.1 Wang's Approach and Black-Scholes

Wang's probability distortion function is based on the standard normal cumulative distribution function. When the underlying security prices are log-normal, Wang's approach results in a very simple form for the price of a contingent claim. This result is proved in the following proposition.

**Proposition 1** *Let  $Z$  be a standard normal variable and  $X$  be a transformation of  $Z$  such that  $X = h(Z)$  where  $h$  is a continuous, positive and increasing function. Then*

$$H(X; \alpha) = \mathbb{E}[h(Z + \alpha)] \quad (\text{KERNEL})$$

where

$$H(X; \alpha) = \int_0^\infty g_\alpha(S_X(t)) dx$$

$$g_\alpha(p) = \Phi[\Phi^{-1}(p) + \alpha]$$

and

$$\Phi(x) = \int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

**Proof.** We have

$$H(X; \alpha) = \int_0^\infty g_\alpha(S_X(t)) dt$$

Now

$$\begin{aligned} S_X(t) &= P[X > t] \\ &= P[h(Z) > t] \\ &= P[Z > h^{-1}(t)] \\ &= 1 - \Phi(h^{-1}(t)) \\ &= \Phi(-h^{-1}(t)) \end{aligned}$$

and

$$\begin{aligned} g_\alpha(S_X(t)) &= \Phi[\Phi^{-1}(\Phi(-h^{-1}(t))) + \alpha] \\ &= \Phi[-h^{-1}(t) + \alpha] \\ &= 1 - \Phi[h^{-1}(t) - \alpha] \\ &= P[Z > h^{-1}(t) - \alpha] \\ &= P[h(Z + \alpha) > t] \end{aligned}$$

Therefore:

$$\begin{aligned} H(X; \alpha) &= \int_0^\infty P[h(Z + \alpha) > t] dt \\ &= \mathbb{E}[h(Z + \alpha)] \end{aligned}$$

The positiveness of  $h$  assumption can be relaxed. The last form of the expression for  $H(X; \alpha)$  follows from a standard result on expected values included in the Appendix. ■

More generally, this proposition can be extended to the case when the security price is a function of a symmetrically distributed random variable.

Indeed, as we will show later, any probability distortion function of the form

$$F(F^{-1}(x) + \alpha)$$

where  $F$  is the cumulative distribution function of the underlying random variable yields a pricing formula with a simple form.

**Pricing a Security** If  $S_t$  is the price of a security at time  $t$ , following a geometric Brownian motion so that  $S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$  where  $W_t$  is a Brownian motion under  $P$  then  $S_T$  can be written as a function of the standard normal random variable  $Z$ . In this case  $S_T = h(Z)$  where  $h(x) = S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma\sqrt{T}x}$ .

Applying the above result (KERNEL) we then have:

$$\begin{aligned} H(S_T; -\alpha) &= \mathbb{E}[h(Z - \alpha)] \\ &= \mathbb{E}[S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma\sqrt{T}Z - \sigma\sqrt{T}\alpha}] \\ &= S_0 e^{(\mu - \frac{\sigma^2}{2})T - \sigma\sqrt{T}\alpha + \frac{\sigma^2 T}{2}} \end{aligned}$$

For  $\alpha = \frac{\mu - r_c}{\sigma} \sqrt{T}$  the expression for  $H(S_T; -\alpha)$  simplifies to:

$$H(S_T; -\alpha) = S_0 e^{r_c T}$$

If we determine the present value of the certainty equivalent using Wang's approach then

the current price of the asset becomes

$$\begin{aligned} e^{-r_c T} H(S_T; -\alpha) &= e^{-r_c T} S_0 e^{r_c T} \\ &= S_0 \end{aligned}$$

Thus the parameter  $\alpha$  calibrates the discounted certainty equivalent of the security price on a future date to the initial price of the security.

**Price of a European Call Option** In the standard Black-Scholes model asset prices follow a geometric Brownian motion with

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW_t$$

so that

$$S_T = S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma W_T}$$

A standard European call option has pay-off at maturity  $T$ :

$$C(T, K) = (S_T - K)^+$$

and we can write this as  $f(Z)$  where  $Z$  is a standard normal random variable and

$$f(Z) = \left( S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma \sqrt{T}Z} - K \right)^+$$

Applying the relation (KERNEL) we have:

$$\begin{aligned} H(C(T, K), \alpha) &= \mathbb{E}[f(Z + \alpha)] \\ &= \int_{-\infty}^{\infty} \left( S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma\sqrt{T}\alpha + \sigma\sqrt{T}z} - K \right)^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \end{aligned}$$

The values of  $z$  for which  $S_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma\sqrt{T}\alpha + \sigma\sqrt{T}z} \geq K$  determines the region of integration.

This region is  $[z_{\min}, \infty)$  where

$$z_{\min} = \frac{\ln(K/S_0) - (\mu - \sigma^2/2)T - \sigma\sqrt{T}\alpha}{\sigma\sqrt{T}}$$

Hence

$$\begin{aligned} H(C(T, K); -\alpha) &= \int_{z_{\min}}^{\infty} \left( S_0 e^{(\mu - \frac{\sigma^2}{2})T - \sigma\sqrt{T}\alpha + \sigma\sqrt{T}z} - K \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ &= \int_{z_{\min}}^{\infty} S_0 e^{(\mu - \frac{\sigma^2}{2})T - \sigma\sqrt{T}\alpha + \sigma\sqrt{T}z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - K \int_{z_{\min}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ &= S_0 e^{\mu T - \sigma\sqrt{T}\alpha} \int_{z_{\min}}^{\infty} e^{-\frac{1}{2}(x - \sigma\sqrt{T})^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - K \int_{z_{\min}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ &= S_0 e^{\mu T - \sigma\sqrt{T}\alpha} [1 - \Phi(z_{\min} - \sigma\sqrt{T})] - K [1 - \Phi(z_{\min})] \\ &= S_0 e^{\mu T - \sigma\sqrt{T}\alpha} \Phi(-z_{\min} + \sigma\sqrt{T}) - K \Phi(-z_{\min}) \end{aligned}$$

Calibrating Wang's discounted certainty equivalent to the underlying security price using

$$\alpha = \frac{(\mu - r_c)}{\sigma} \sqrt{T}$$

gives

$$e^{-r_c T} H(C(T, K); -\alpha) = S_0 \Phi \left( \frac{\ln(\frac{S_0}{K}) + (r_c + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \right) - e^{-r_c T} K \Phi \left( \frac{\ln(\frac{S_0}{K}) + (r_c - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \right)$$

which is the Black-Scholes price of the call option at time 0.

This demonstrates that Wang's probability distortion function approach recovers the Black-Scholes price of a European call option with  $\alpha_i = \frac{\mu_i - r_c}{\sigma_i} \sqrt{T}$ .

## 5.2 Probability Distortion Approach and Change of Measure

Given that Wang's approach using a probability distortion operator is consistent with financial theory when the underlying security price is driven by a geometric Brownian motion, it should be possible to show that it is equivalent to a change of measure when determining the expected pay-off of the security under a risk neutral probability measure.

For the Black-Scholes case where  $S_t$  has a log-normal distribution, the dynamics of  $S_t$  are given by the stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dW_t)$$

under the "real-world" probability measure  $P$ . The solution to this SDE is

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$$

The risk neutral probability measure  $Q$ , under which the discounted security prices are a martingale is determined by the likelihood ratio, or Radon-Nikodym derivative, given by

$$m(t) = \frac{dQ}{dP} \Big|_{\mathcal{F}_t} = \exp \left\{ - \left( \frac{\mu - r}{\sigma} \right) W_t - \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 t \right\}$$

such that

$$\mathbb{E}_Q[X] = \mathbb{E}[mX]$$

By the Girsanov theorem, the process

$$W_t^* = W_t + \frac{\mu - r}{\sigma}t$$

is a standard Brownian motion under  $Q$  and

$$dW_t^* = dW_t + \frac{\mu - r}{\sigma}dt$$

Writing the dynamics of  $S$  under  $Q$  gives:

$$dS_t = S_t(rdt + \sigma dW_t^*)$$

so that

$$S_t = S_0 e^{(r - \frac{\sigma^2}{2})t + \sigma W_t^*}$$

The effect of the change of probability measure is to shift the drift from  $\mu$  to  $r$ . The process  $m(t)$  is the stochastic discount factor and in the Black-Scholes case we have an explicit form for this discount factor. In fact it has a log-normal distribution.

Now

$$\begin{aligned} P[S_t \leq x] &= P \left[ \ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t \leq \ln x \right] \\ &= \Phi \left( \frac{\ln(x/S_0) - \left(\mu - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} \right) \end{aligned}$$

Under the risk-neutral measure

$$Q[S_t \leq x] = \Phi \left( \frac{\ln(x/S_0) - \left(r - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} \right)$$

We then have

$$\begin{aligned} \frac{\ln(x/S_0) - (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} &= \frac{\ln(x/S_0) - (\mu - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} + \frac{\mu - r}{\sigma}\sqrt{t} \\ &= \Phi^{-1}(P[S_t \leq x]) + \frac{\mu - r}{\sigma}\sqrt{t} \end{aligned}$$

so that

$$Q[S_t \leq x] = \Phi\left(\Phi^{-1}(P[S_t \leq x]) + \frac{\mu - r}{\sigma}\sqrt{t}\right)$$

This representation is used in credit risk modeling where the default probability under the “real world” probability distribution is related to the default probability under the risk neutral distribution used for pricing (see Bohn 1999 [2]).

In the log-normal case the intuition behind using Wang’s approach is clear since the probability under the  $Q$  measure is the same as that given by Wang’s probability distortion function.

We can see this since

$$\begin{aligned} Q[S_t > x] &= 1 - Q[S_t \leq x] \\ &= 1 - \Phi\left(\Phi^{-1}(P[S_t \leq x]) + \frac{\mu - r}{\sigma}\sqrt{t}\right) \\ &= \Phi\left(-\Phi^{-1}(P[S_t \leq x]) - \frac{\mu - r}{\sigma}\sqrt{t}\right) \end{aligned}$$

and

$$-\Phi^{-1}(y) = \Phi^{-1}(1 - y)$$

So

$$\begin{aligned}
Q[S_t > x] &= \Phi\left(\Phi^{-1}(1 - P[S_t \leq x]) - \frac{\mu - r}{\sigma}\sqrt{t}\right) \\
&= \Phi\left(\Phi^{-1}(P[S_t > x]) - \frac{\mu - r}{\sigma}\sqrt{t}\right)
\end{aligned} \tag{2}$$

Now, for any attainable contingent claim of the form  $X_T = h(S_T)$ , where  $h$  is a positive increasing function we have

$$\begin{aligned}
Q[X_T > x] &= Q[h(S_T) > x] \\
&= Q[S_T > h^{-1}(x)] \\
&= \Phi\left(\Phi^{-1}(P[S_T > h^{-1}(x)]) - \frac{\mu - r}{\sigma}\sqrt{T}\right) \\
&= \Phi\left(\Phi^{-1}(P[h(S_T) > x]) - \frac{\mu - r}{\sigma}\sqrt{T}\right) \\
&= \Phi\left(\Phi^{-1}(P[X_T > x]) - \frac{\mu - r}{\sigma}\sqrt{T}\right)
\end{aligned}$$

Since expectation under the measure  $Q$  is given by

$$\mathbb{E}_Q[X_T] = \int_0^\infty Q[X_T > x] dx$$

for a positive random variable  $X_T$ . We obtain

$$\begin{aligned}
\mathbb{E}_Q[X_T] &= \int_0^\infty \Phi\left(\Phi^{-1}(P[X_T > x]) - \frac{\mu - r}{\sigma}\sqrt{t}\right) dx \\
&= H(X_T; -\frac{\mu - r}{\sigma}\sqrt{t})
\end{aligned}$$

## 6 Generalisation of Wang's Approach

### 6.1 Symmetric Distributions

We can generalise the previous proposition where the underlying variable  $Z$  has a normal probability distribution to the case where  $Z$  is symmetrically distributed.

**Proposition 2** *Let  $Z$  be a random variable with a cumulative distribution function  $F$  whose probability density function is symmetric around 0. And let  $X$  be a transformation of  $Z$  such that  $X = h(Z)$  where  $h$  is a continuous, positive and increasing function. Define the operator*

$$I(X, \alpha) = \int_0^\infty F[F^{-1}(\Pr[X > t]) + \alpha] dt$$

*We have*

$$I(X, \alpha) = \mathbb{E}[h(Z + \alpha)]$$

**Proof.** For any  $F$  whose probability density is symmetric around 0 we will have  $F(-x) = 1 - F(x)$ . For  $X = h(Z)$ :

$$\begin{aligned} S_X(t) &= P[X > t] \\ &= P[h(Z) > t] \\ &= P[Z > h^{-1}(t)] \\ &= 1 - F(h^{-1}(t)) \\ &= F(-h^{-1}(t)) \end{aligned}$$

Define  $g_\alpha(S_X(t)) = F[F^{-1}(S_X(t)) + \alpha]$ , then

$$\begin{aligned}
g_\alpha(S_X(t)) &= F[F^{-1}(F(-h^{-1}(t))) + \alpha] \\
&= F[-h^{-1}(t) + \alpha] \\
&= 1 - F[h^{-1}(t) - \alpha] \\
&= P[Z > h^{-1}(t) - \alpha] \\
&= P[h(Z + \alpha) > t]
\end{aligned}$$

Therefore:

$$\begin{aligned}
I(X, \alpha) &= \int_0^\infty g_\alpha(S_X(t)) dt \\
&= \int_0^\infty P[h(Z + \alpha) > t] dt \\
&= \mathbb{E}[h(Z + \alpha)]
\end{aligned}$$

■

This is a general expression for  $H(X; \alpha)$  when  $X = h(Z)$  is a function of a random variable  $Z$  whose probability density function is symmetric around 0. Since  $Z = h^{-1}(X)$ , we have

$$I(X, \alpha) = \mathbb{E}[h(h^{-1}(X) + \alpha)]$$

## 6.2 Time-varying Drift and Volatility for the Security Price

We will consider the case where the underlying security price is log-normally distributed with time-varying drift and volatility and show how Wang's approach can be extended in order to derive prices consistent with financial theory.

Let  $X_t$  be a state variable following the “real-world” dynamics

$$dX_t = \mu(t)dt + \sigma(t)dW_t \quad (3)$$

where the drift and the volatility  $\mu(t)$  and  $\sigma(t)$  are functions of time. Formally we have

$$X_t = X_0 + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW_s$$

Assume that the underlying security price

$$S_t = e^{X_t}$$

so that  $S_t$  is log-normally distributed. The dynamics of  $S_t$  are given by Itô’s formula as

$$dS_t = S_t \left( \mu(t) + \frac{1}{2}\sigma^2(t) \right) dt + S_t\sigma(t)dW_t \quad (4)$$

It is straightforward to relate the real world probabilities to the risk neutral probabilities. In fact, for a positive real number  $x$ , we have

$$\begin{aligned} P[S_t \leq x] &= P[X_t \leq \ln(x)] \\ &= \Phi \left( \frac{\ln(x) - \tilde{\mu}_t}{\tilde{\sigma}_t} \right) \end{aligned} \quad (5)$$

where

$$\begin{aligned} \tilde{\mu}_t &= X_0 + \int_0^t \mu(s)ds \\ \tilde{\sigma}_t^2 &= \int_0^t \sigma^2(s)ds \end{aligned}$$

and  $\Phi$  is the standard normal cumulative distribution function.

Denote the market price of risk for the security  $S_t$  by  $\gamma_t$ . The form of  $\gamma_t$  is determined by the fact that the security price, discounted to present value at the risk free interest rate, will be a martingale under the risk neutral probability measure. The market price of risk in an arbitrage-free economy is given by

$$\gamma_t = \frac{\mu(t) + \frac{\sigma(t)^2}{2} - r_c}{\sigma(t)} \quad (6)$$

where  $r_c$  is the risk free interest rate assumed to be constant. By the Girsanov theorem (see Cox and Huang, 1989, [4, Lemma 2.1]), the process

$$W_t^* = W_t + \int_0^t \gamma_s ds$$

is a  $Q$ -Brownian motion, where  $Q$  is the risk neutral probability measure. The above representation is equivalent to

$$dW_t^* = dW_t + \gamma_t dt$$

The dynamics of  $X_t$  under the risk neutral probability will therefore be

$$dX_t = [\mu(t) - \sigma(t)\gamma_t] dt + \sigma(t)dW_t^*$$

or

$$X_t = X_0 + \int_0^t [\mu(s) - \sigma(s)\gamma_s] ds + \int_0^t \sigma(s)dW_s^*$$

For  $x > 0$ , the probability distribution of  $S_t$  under  $Q$  is given by

$$\begin{aligned}
Q[S_t \leq x] &= Q[X_t \leq \ln(x)] \\
&= Q\left[X_0 + \int_0^t (\mu(s) - \sigma(s)\gamma_s)ds + \int_0^t \sigma(s)dW_s^* \leq \ln(x)\right] \\
&= Q\left[X_0 + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW_s^* \leq \ln(x) + \int_0^t \sigma(s)\gamma_s ds\right]
\end{aligned}$$

The volatility  $\sigma(t)$  is deterministic, so

$$\int_0^t \sigma(s)dW_s \stackrel{P}{\sim} N\left(0, \int_0^t \sigma^2(s)ds\right)$$

and

$$\int_0^t \sigma(s)dW_s^* \stackrel{Q}{\sim} N\left(0, \int_0^t \sigma^2(s)ds\right)$$

since  $W_t^*$  is a  $Q$ -Brownian motion. Moreover,  $X_0$  and  $\mu(t)$  are also assumed to be deterministic,

so

$$Q\left[X_0 + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW_s^* \leq y(t)\right] = P\left[X_0 + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW_s \leq y(t)\right]$$

for any deterministic function  $y$ . It follows then that

$$\begin{aligned}
Q[S_t \leq x] &= P\left[X_t \leq \ln(x) + \int_0^t \sigma(s)\gamma_s ds\right] \\
&= \Phi\left(\frac{\ln(x) + \int_0^t \sigma(s)\gamma_s ds - \tilde{\mu}_t}{\tilde{\sigma}_t}\right)
\end{aligned}$$

From equation (5) we have

$$\ln(x) = \tilde{\mu}_t + \tilde{\sigma}_t \Phi^{-1}(P[S_t \leq x])$$

Hence

$$Q[S_t \leq x] = \Phi \left( \Phi^{-1}(P[S_t \leq x]) + \frac{\int_0^t \sigma(s) \gamma_s ds}{\sqrt{\int_0^t \sigma^2(s) ds}} \right) \quad (7)$$

The above equation expresses the relationship between the cumulative distributions of the security price under the real-world and the risk-neutral probabilities. Using the symmetry of  $\Phi$  we also have

$$Q[S_t > x] = \Phi \left( \Phi^{-1}(P[S_t > x]) - \frac{\int_0^t \sigma(s) \gamma_s ds}{\sqrt{\int_0^t \sigma^2(s) ds}} \right) \quad (8)$$

For a formal proof of (8), see the appendix. Let us define the following probability distortion function

$$g_\gamma(u) = \Phi \left( \Phi^{-1}(u) + \frac{\int_0^t \sigma(s) \gamma_s ds}{\sqrt{\int_0^t \sigma^2(s) ds}} \right)$$

and define the certainty equivalent as the following

$$H(Y; \gamma) = \int_0^\infty g_\gamma(P[Y > u]) du$$

for any non-negative random variable  $Y$ .

From the equation (8) and the definition of the distortion function  $g_\gamma$  we have

$$Q[S_t > x] = g_{-\gamma}(P[S_t > x])$$

Now, since  $\mathbb{E}_Q[S_t] = \int_0^\infty Q[S_t > x] dx$ , it follows that

$$\mathbb{E}_Q[S_t] = H(S_t; -\gamma)$$

Hence, the discounted certainty equivalent of  $S_t$  is equal to the arbitrage-free price of the security.

Let us consider the case of any attainable contingent claim of the form  $X_T = h(S_T)$ , where  $h$  is a positive increasing function we have

$$\begin{aligned} Q[X_T > x] &= Q[h(S_T) > x] \\ &= Q[S_T > h^{-1}(x)] \\ &= g_{-\gamma}(P[S_T > h^{-1}(x)]) \\ &= g_{-\gamma}(P[X_T > x]) \end{aligned}$$

Integrating both sides from 0 to infinity yields

$$\begin{aligned} \mathbb{E}_Q[X_T] &= \int_0^\infty g_{-\gamma}(P[X_T > x]) dx \\ &= H(X_T; -\gamma) \end{aligned}$$

Here again, the arbitrage-free price of any contingent claim of the form  $X_T = h(S_T)$  where  $h$  is a positive increasing function, is equal to  $e^{-rT}H(X_T; -\gamma)$ . The advantage of such representation is that arbitrage-free prices can be computed straightforwardly from the distribution of the underlying risk and the market price of risk  $\gamma$ .

## 7 Applications

Wang's approach is readily implemented using simulation. For general underlying security price distributions it is not possible to derive an analytical formula for the certainty equivalent. In this case it is necessary to implement contingent claim valuation using simulation. This is often the case in practice where empirical studies suggest that security prices are not log-normal.

To illustrate the procedure we consider the valuation of European call options. The results are based on a simulation of 1000 log-normal prices with drift  $\mu = 16\%$  and volatility  $\sigma = 20\%$ . The risk free interest rate  $r = 8\%$  and the current market price of the underlying security is assumed to be \$20. The maturity of the call is 0.5 years.

The simulated security prices are used to estimate  $\alpha = \frac{\mu_i - r_c}{\sigma_i} \sqrt{T}$  such that the discounted certainty equivalent using the simulated future security prices, and Wang's normal based probability distortion function, will equal the initial security price. We then obtain  $\hat{\alpha} = 0.283989$ . This value for  $\alpha$  is used to then price 3 calls with strikes \$18, \$20, \$22. Note that the actual value of  $\alpha = 0.282843$  based on the assumptions used to simulate the security prices.

The following values were obtained and are compared with the theoretical Black-Scholes price that would have been determined using the analytical formula.

Insert table (1) about here

This shows that the calibration process using simulated values is reasonably accurate for these calls and suggests that pricing contingent claims with simulation and the certainty equivalent is numerically reliable.

So far the case of log-normal security prices has been mainly considered. It is possible to apply this approach to other security price distributions. In order to do this we apply a similar algorithm to that used above. This is done for both the commonly used binomial model and also for the constant elasticity of variance model. We first calibrate the certainty equivalent for the future security price to the initial price in order to determine the parameter  $\alpha$  to be used

in contingent claim pricing. This calibrated parameter is then used to evaluate the certainty equivalent using the normal based probability distortion function of Wang for contingent claims.

## 7.1 The Cox Ross and Rubinstein binomial model

We compare risk neutral prices determined exactly and Wang's certainty equivalent prices for a European call option using a binomial lattice model for the underlying security (see Figure (6)).

Insert figure (6) about here

The value of a call option with pay-off  $X_T = (S_T - K)^+$  where  $S_T$  is the price of the underlying security at time  $T$  is given by

$$E_Q[X_T] = \sum_{k=0}^T Q(\omega_k) X_T(\omega_k)$$

where

$$Q(\omega_k) = \binom{T}{k} q^k (1-q)^{T-k}$$

and  $q$  is the risk neutral probability of the security up jump in the lattice.

Hamada, Sherris and van der Hoek (2001, [6]) establish the discrete-time analogue to the expression of  $H[X; \alpha]$ . It is given by

$$H[X_T; \alpha] = \sum_{k=0}^T \{g_\alpha(P[X \geq X(\omega_k)]) - g_\alpha(P[X > X(\omega_k)])\} X_T(\omega_k)$$

In theory we may be able to determine an explicit expression for  $\alpha$  as follows. Let  $S(k)$  denote the function  $P[X \geq X(\omega_k)]$ . We have

$$S(k) = \sum_{i=k}^T \binom{T}{i} p^i (1-p)^{T-i}$$

If  $H[X_T; \alpha] = E_Q[X_T]$  for any contingent claim, then this is true for Arrow-Debreu securities paying 1 at some state  $\omega$  and 0 elsewhere. This implies that

$$Q(\omega) = g_\alpha(P[X \geq X(\omega)]) - g_\alpha(P[X > X(\omega)]), \quad \forall \omega \in -$$

or

$$\left\{ \begin{array}{l} \binom{T}{k} q^k (1-q)^{T-k} = g_\alpha[S(k)] - g_\alpha[S(k+1)] \text{ for all } k = 0 \text{ to } T-1 \\ q^T = g_\alpha[p^T] \end{array} \right.$$

There is clearly no value for  $\alpha$  that will satisfy all of these  $T+1$  equations simultaneously. However we can estimate an  $\alpha$  by determining the certainty equivalent of the security price distribution and then calibrating the  $\alpha$  so that the discounted certainty equivalent is equal to the current price. This parameter estimate is then used to value contingent claims on this security.

The model for the security price is the binomial model with a continuous compounding expected return of  $\mu = 20\%$  and a volatility of  $\sigma = 20\%$ . The risk-free interest rate is  $r = 6\%$ , the time step is  $\Delta T = \frac{1}{12}$  of a year. For these assumptions, the real world and the risk neutral probabilities of an up jump in the binomial lattice are  $p = \frac{e^{\mu\Delta T} - d}{u - d} = 0.631$  and  $q = \frac{e^{r\Delta T} - d}{u - d} = 0.529$  respectively. The size of an up jump is  $u = e^{\sigma\sqrt{\Delta T}}$  and the down jump is  $d = \frac{1}{u}$ . This produces an arbitrage-free lattice for the risk neutral security price distribution.

The security price distribution at the end of a year is shown in the Table below. The columns in the Table show the real world security price distribution  $p(x)$ , the decumulative security price distribution  $P[X > x]$ , the risk neutral security price distribution  $q(x)$ , the distorted decumulative real world probability distribution using the normal probability distortion function of Wang  $g(P[X > x])$  and the distorted probability security price distribution corresponding to Wang's distortion function  $q^*(x)$ . In order to derive the distorted probabilities it was necessary to solve iteratively for the  $\alpha$  that calibrated the discounted certainty equivalent of the security price at the end of the year to the initial security price of \$100. The parameter that calibrated the price distribution was  $\alpha = 0.73102$ . Based on the equivalent continuous time security price process the value is  $\alpha = \frac{0.2-0.06}{0.2} = 0.7$ .

Insert table (2) about here

In order to illustrate the pricing of a contingent claim we price a call option on the security with strike \$105. The Table below shows the pay-off at the end of the year for the call option along with the pay-off weighted by the risk neutral probabilities and also the distorted probabilities.

Insert table (3) about here

The risk neutral price of the call is 8.623 and Wang's distorted probability price is 8.560, a difference of 0.063 or 0.7%. Since the binomial lattice converges to the log-normal distribution in the limit, Wang's distorted probability price will also converge to the Black-Scholes price as does the risk neutral price determined from the lattice. The two prices will however differ for relatively small numbers of steps in the lattice.

Although the distorted probability approach can be implemented in a binomial lattice for European options, it is not as easy to implement for options with early exercise.

## 7.2 The CEV model

As another example of underlying security prices that are not lognormal, we consider the constant elasticity of variance model introduced by Cox and Ross (1976 [5]). The dynamics of the underlying security prices are given by

$$dS_t = \mu S_t dt + \sigma S_t^{\beta/2} dZ$$

where  $dZ$  is an increment of Weiner process. The elasticity of return variance with respect to price equals  $\beta - 2$ , and if  $\beta < 2$ , volatility and prices are inversely related. If  $\beta = 2$  then elasticity is zero and prices are lognormally distributed with the variance of returns constant, as assumed in the Black-Scholes model. Beckers (1980 [1]) estimates  $\beta$  for forty seven stocks and concludes that the CEV diffusion process “could be a better descriptor of the actual stock price behaviour than the traditionally used lognormal model”.

This model is used as a more realistic alternative for the underlying security dynamics, where the prices exhibit skewness and kurtosis, which is also typically the case for insurance claims. Cox and Ross (1976, [5]) show that for a CEV process with  $\beta$  less than two, the density function of the stock price at time  $T$  conditional on the current stock price in a risk neutral world (with  $\mu = r - a$ ) is given by

$$f(S_T, T | S_t, t) = (2 - \beta) k^{\frac{1}{2-\beta}} (x\omega^{1-2\beta})^{\frac{1}{(4-2\beta)}} e^{-x-\omega} I_{\frac{1}{(2-\beta)}}(2\sqrt{x\omega})$$

where

$$k = \frac{2(r - a)}{\sigma^2(2 - \beta) [e^{(r-a)(2-\beta)\tau} - 1]}$$

$$x = kS_t^{2-\beta} e^{(r-a)(2-\beta)\tau}$$

$$\omega = kS_T^{2-\beta}$$

and  $\tau = T - t$ ;  $I_q(\cdot)$  is the modified Bessel function of the first kind of order  $q$ ;  $r$  denotes the risk free interest rate and  $a$  denotes the continuous proportional dividend rate.

Schroder (1989, [13]) obtains a closed form solution to a European call option when the underlying security dynamics are given by the CEV model. The price is given as a function of the non-central Chi-square cumulative function. If  $C$  is the price of the call with exercise price  $E$ , then

$$C = S_t e^{-a\tau} \chi \left( 2y; 2 + \frac{2}{2-\beta}, 2x \right) - E e^{-r\tau} \left[ 1 - \chi \left( 2x; \frac{2}{2-\beta}, 2y \right) \right] \quad (9)$$

where  $y = kE^{2-\beta}$  and  $\chi(z; d, x)$  is the non-central Chi-square *decumulative* distribution function evaluated at  $z$ , with  $d$  degrees of freedom and non-central parameter  $x$ .

We will price a number of call options using Wang's probability distortion function approach and compare the prices with the risk neutral arbitrage-free price given by (9). We will do this by simulating 2000 security prices for the CEV model at the end of a year using a discrete time approximation to the diffusion process with a time step of 1/100. The real world security prices are simulated with  $\mu = 15\%$  and we use  $r = 5\%$  and volatility  $\sigma = 20\%$ . The current market price of the underlying security is \$20. The maturity of the call is 1 year.

Once again the  $\alpha$  parameter is determined from the simulated prices at the end of the year to match the initial security price. The initial value of  $\alpha$ , equal to 2.2497 is tweaked to the value  $\hat{\alpha} = 2.1419$ . This value is then used to price 6 call options on the security with strikes from \$18

to \$23.

The Table below shows the difference between the arbitrage-free price and the option price using Wang's probability distortion.

Insert table (4) about here

In order to illustrate the effect of the departure from log-normality on the pricing we have also valued the options using the Black-Scholes model and the simulation approach to implement Wang's probability distortion function approach. The results are given below.

Insert table (5) about here

Unlike the log-normal model, where Wang's probability distortion function approach only differs from the risk neutral exact price due to the errors introduced by the discretisation and simulation, in the CEV model the difference between the prices is increasing as the option goes further out-of-the money. This results from the skewness and kurtosis of the underlying security price distribution differing significantly from the log-normal security price distribution.

The reason for this can be understood by noting that the effect of the  $\alpha$  parameter is to shift only the mean of the distribution but it does not adjust for the skewness or the kurtosis. This is illustrated by Figure (7). This example demonstrates that applying Wang's probability distortion function approach using the normal probability transform to price contingent claims where the underlying security dynamics are not lognormal will result in substantial errors.

Insert figure (7) about here

## 8 Conclusion

This paper has considered a recently developed approach proposed for contingent claims pricing originally developed for pricing insurance risk. The approach is consistent with a dual theory

to expected utility theory which is most often used as the basis of financial theory including equilibrium pricing of contingent claims. We formally prove that the approach will reproduce the Black-Scholes option pricing formula assuming that security prices are log-normal. We extend the approach to the case where the underlying security or risk has a symmetric distribution, and also consider the log-normal case with time varying parameters and formally derive the relationship between the parameter used in the probability distortion function approach and the market price of risk for the underlying security. Finally we highlight the limitations of applying the approach using the normal probability distortion function as proposed by Wang and illustrate the potential issues in implementing the technique using the binomial lattice model and the CEV model for security prices.

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Appendix - Derivation of results

- $\mathbb{E}[X] = \int_{-\infty}^0 [S_X(x) - 1]dx + \int_0^{\infty} S_X(x)dx$

- if  $X \geq 0$ , then  $X = \int_0^{\infty} I_{\{X > u\}} du$ . Using Fubini theorem,  $\mathbb{E}[X] = \mathbb{E}[\int_0^{\infty} I_{\{X > u\}} du] = \int_0^{\infty} \mathbb{E}I_{\{X > u\}} du = \int_0^{\infty} P[X > u] du$

- if  $X \leq 0$ , then  $X = \int_{-\infty}^0 -I_{\{X < u\}} du$ . ,  $\mathbb{E}[X] = \mathbb{E}[\int_{-\infty}^0 -I_{\{X < u\}} du] = - \int_{-\infty}^0 \mathbb{E}[I_{\{X < u\}}] du = - \int_{-\infty}^0 P[X < u] du = \int_{-\infty}^0 \{P[X > u] - 1\} du$

So in the general case:  $\mathbb{E}[X] = \int_{-\infty}^0 \{P[X > u] - 1\} du + \int_0^{\infty} P[X > u] du$ .

- For  $X \sim N(\mu, \sigma^2)$  we have

$$\begin{aligned} S_X(t) &= P[X > t] \\ &= 1 - P\left[\frac{X - \mu}{\sigma} < \frac{t - \mu}{\sigma}\right] \\ &= 1 - \Phi_{0,1}\left(\frac{t - \mu}{\sigma}\right) \end{aligned}$$

Wang's distorted decumulative distribution function is:

$$\begin{aligned} g_{\alpha}[S_X(t)] &= \Phi_{0,1}\left[\Phi_{0,1}^{-1}(S_X(t)) + \alpha\right] \\ &= \Phi_{0,1}\left[\Phi_{0,1}^{-1}\left\{1 - \Phi_{0,1}\left(\frac{t - \mu}{\sigma}\right)\right\} + \alpha\right] \\ &= \Phi_{0,1}\left[\Phi_{0,1}^{-1}\left\{\Phi_{0,1}\left(-\frac{t - \mu}{\sigma}\right)\right\} + \alpha\right] \quad \text{by symmetry of the density} \\ &= \Phi_{0,1}\left[\left(-\frac{t - \mu - \alpha\sigma}{\sigma}\right)\right] \quad \text{by Normal distribution assumption} \\ &= 1 - \Phi_{0,1}\left[\left(\frac{t - (\mu + \alpha\sigma)}{\sigma}\right)\right] \quad \text{also by symmetry of the density} \\ &= S_Z(t) \quad \text{where } Z \sim N(\mu + \alpha\sigma, \sigma^2) \end{aligned}$$

and

$$\begin{aligned}
H[X; \alpha] &= \int_{-\infty}^0 \{g_\alpha[S_X(t)] - 1\} dt + \int_0^\infty g_\alpha[S_X(t)] dt \\
&= \int_{-\infty}^0 \{S_Z(t) - 1\} dt + \int_0^\infty S_Z(t) dt \\
&= \mathbb{E}(Z) \\
&= \mu + \alpha\sigma
\end{aligned}$$

- For  $Y = \ln X \sim N(\mu, \sigma^2)$ , then

$$\begin{aligned}
S_Y(t) &= P[Y > t] \\
&= 1 - P\left[\frac{\ln Y - \mu}{\sigma} < \frac{\ln t - \mu}{\sigma}\right] \\
&= 1 - \Phi_{0,1}\left(\frac{\ln t - \mu}{\sigma}\right)
\end{aligned}$$

Wang's distorted decumulative distribution function is:

$$\begin{aligned}
g_\alpha[S_Y(t)] &= \Phi_{0,1}\left[\Phi_{0,1}^{-1}(S_Y(t)) + \alpha\right] \\
&= \Phi_{0,1}\left[\Phi_{0,1}^{-1}\left\{1 - \Phi_{0,1}\left(\frac{\ln t - \mu}{\sigma}\right)\right\} + \alpha\right] \\
&= \Phi_{0,1}\left[\Phi_{0,1}^{-1}\left\{\Phi_{0,1}\left(-\frac{\ln t - \mu}{\sigma}\right)\right\} + \alpha\right] \quad \text{by symmetry of the Normal density} \\
&= \Phi_{0,1}\left[\left(-\frac{\ln t - \mu - \alpha\sigma}{\sigma}\right)\right] \\
&= 1 - \Phi_{0,1}\left[\left(\frac{\ln t - (\mu + \alpha\sigma)}{\sigma}\right)\right] \quad \text{also by symmetry of the Normal density} \\
&= S_Z(u) \quad \text{where } \ln Z \sim N(\mu + \alpha\sigma, \sigma^2)
\end{aligned}$$

We then have

$$\begin{aligned}
H[X; \alpha] &= \int_0^\infty g_\alpha[S_X(t)] dt \\
&= \int_0^\infty S_Z(u) dt \\
&= \mathbb{E}(Z) \\
&= e^{\mu + \alpha\sigma + \frac{\sigma^2}{2}}
\end{aligned}$$

- For  $\ln Z \sim N\left(\ln X_i(0) + [\mu_i - \frac{1}{2}\sigma_i^2]T - \alpha_i\sigma_i\sqrt{T}, \sigma_i^2T\right)$

$$\begin{aligned}
&\int_K^\infty S_Z(t) dt \\
&= e^{-r_c T} \int_0^\infty \left[ 1 - \Phi_{0,1}\left(\frac{\ln(K+u) - \{\ln X_i(0) + [\mu_i - \frac{1}{2}\sigma_i^2]T\} + \alpha_i\sigma_i\sqrt{T}}{\sigma_i\sqrt{T}}\right) \right] du \\
&= e^{-r_c T} \int_{\ln K}^\infty e^t \left[ 1 - \Phi_{0,1}\left(\frac{t - \{\ln X_i(0) + [\mu_i - \frac{1}{2}\sigma_i^2]T\} + \alpha_i\sigma_i\sqrt{T}}{\sigma_i\sqrt{T}}\right) \right] dt \\
&= e^{-r_c T} \left[ - \left[ e^{\ln K} (1 - \Phi_{0,1}\left(\frac{\ln K - \{\ln X_i(0) + [\mu_i - \frac{1}{2}\sigma_i^2]T\} + \alpha_i\sigma_i\sqrt{T}}{\sigma_i\sqrt{T}}\right)) \right] \right. \\
&\quad \left. + \int_{\ln K}^\infty e^t \frac{d}{du} \left[ \Phi_{0,1}\left(\frac{t - \{\ln X_i(0) + [\mu_i - \frac{1}{2}\sigma_i^2]T\} + \alpha_i\sigma_i\sqrt{T}}{\sigma_i\sqrt{T}}\right) \right] du \right] \\
&= e^{-r_c T} \left[ -K \left[ (1 - \Phi_{0,1}\left(\frac{\ln K - \{\ln X_i(0) + [\mu_i - \frac{1}{2}\sigma_i^2]T\} + \alpha_i\sigma_i\sqrt{T}}{\sigma_i\sqrt{T}}\right)) \right] \right. \\
&\quad \left. + e^{\mu_i T + \alpha_i\sigma_i\sqrt{T}} \left[ 1 - \Phi_{0,1}\left(\frac{\ln K - \{\ln X_i(0) + [\mu_i + \frac{1}{2}\sigma_i^2]T\} + \alpha_i\sigma_i\sqrt{T}}{\sigma_i\sqrt{T}}\right) \right] \right] du \\
&= e^{-r_c T} \left[ -K \left[ \Phi_{0,1}\left(-\frac{\{\ln \frac{X_i(0)}{K} - [\mu_i - \frac{1}{2}\sigma_i^2]T\}}{\sigma_i\sqrt{T}} - \alpha_i\right) \right] \right. \\
&\quad \left. + e^{\mu_i T - \alpha_i\sigma_i\sqrt{T}} \left[ \Phi_{0,1}\left(-\frac{\{\ln \frac{X_i(0)}{K} - [\mu_i + \frac{1}{2}\sigma_i^2]T\}}{\sigma_i\sqrt{T}} - \alpha_i\right) \right] \right] du \\
&= \left[ \begin{array}{c} X_i(0)e^{(\mu_i - r_c)T - \alpha_i\sigma_i\sqrt{T}} N(d_1) \\ -Ke^{-r_c T} N(d_2) \end{array} \right]
\end{aligned}$$

where

$$N(d_1) = \Phi \left( \frac{\ln\left(\frac{X_i(0)}{K}\right) + \left(\mu_i + \frac{\sigma_i^2}{2}\right)T}{\sigma_i \sqrt{T}} - \alpha_i \right)$$

$$N(d_2) = \Phi \left( \frac{\ln\left(\frac{X_i(0)}{K}\right) + \left(\mu_i - \frac{\sigma_i^2}{2}\right)T}{\sigma_i \sqrt{T}} - \alpha_i \right)$$

- Proof of  $Q[S_t > x] = g_{-\gamma}(P[S_t > x])$

$$\begin{aligned} Q[S_t \leq x] &= g_\gamma(P[S_t \leq x]) \\ &= \Phi \left( \Phi^{-1}(P[S_t \leq x]) + \frac{\int_0^t \sigma(s) \gamma_s ds}{\sqrt{\int_0^t \sigma^2(s) ds}} \right) \end{aligned}$$

So

$$\begin{aligned} Q[S_t > x] &= 1 - Q[S_t \leq x] \\ &= 1 - \Phi \left( \Phi^{-1}(P[S_t \leq x]) + \frac{\int_0^t \sigma(s) \gamma_s ds}{\sqrt{\int_0^t \sigma^2(s) ds}} \right) \\ &= 1 - \Phi \left( \Phi^{-1}(1 - P[S_t > x]) + \frac{\int_0^t \sigma(s) \gamma_s ds}{\sqrt{\int_0^t \sigma^2(s) ds}} \right) \end{aligned}$$

Using the fact that  $\Phi^{-1}(1 - u) = -\Phi^{-1}(u)$ , and  $1 - \Phi(u) = \Phi(-u)$ , the above expression

simplifies to

$$\begin{aligned} Q[S_t > x] &= 1 - \Phi \left( -\Phi^{-1}(P[S_t > x]) + \frac{\int_0^t \sigma(s) \gamma_s ds}{\sqrt{\int_0^t \sigma^2(s) ds}} \right) \\ &= \Phi \left( \Phi^{-1}(P[S_t > x]) - \frac{\int_0^t \sigma(s) \gamma_s ds}{\sqrt{\int_0^t \sigma^2(s) ds}} \right) \\ &= g_{-\gamma}(P[S_t > x]) \end{aligned}$$

<b>strike price</b>	18.000	20.000	22.000
<b>B-S price</b>	2.913	1.541	0.678
<b>Wang's price</b>	2.927	1.563	0.707

Figure 1: Call values comparison between Black-Scholes and Wang's prices in the lognormal case

Node	S(T)	p(x)	P[X>x]	q(x)	g (P[X>x] )	q*(x)
0	199.93	0.0040	0.0000	0.0005	0.0000	0.0004
1	178.13	0.0280	0.0040	0.0051	0.0004	0.0045
2	158.71	0.0900	0.0320	0.0251	0.0049	0.0241
3	141.40	0.1753	0.1219	0.0745	0.0290	0.0743
4	125.98	0.2307	0.2973	0.1494	0.1032	0.1511
5	112.24	0.2158	0.5279	0.2128	0.2543	0.2153
6	100.00	0.1472	0.7437	0.2211	0.4696	0.2219
7	89.09	0.0738	0.8909	0.1688	0.6916	0.1676
8	79.38	0.0270	0.9647	0.0939	0.8592	0.0925
9	70.72	0.0070	0.9916	0.0372	0.9517	0.0366
10	63.01	0.0012	0.9986	0.0099	0.9883	0.0099
11	56.14	0.0001	0.9999	0.0016	0.9982	0.0017
12	50.02	0.0000	1.0000	0.0001	0.9999	0.0001

Figure 2: Binomial Lattice - Security Price Probability Distributions

payoff $Y(x)$	$Y(x)q(x)$	$Y(x)q^*(x)$
94.93	0.046	0.034
73.13	0.375	0.331
53.71	1.349	1.292
36.40	2.713	2.704
20.98	3.133	3.169
7.24	1.541	1.559
0.00	0.000	0.000
0.00	0.000	0.000
0.00	0.000	0.000
0.00	0.000	0.000
0.00	0.000	0.000
0.00	0.000	0.000
0.00	0.000	0.000
0.00	0.000	0.000

Figure 3: Pricing a Call Option with Strike 105

<b>strike price</b>	18.000	19.000	20.000	21.000	22.000	23.000
<b>Arbitrage-free price</b>	2.8779	1.9304	1.0332	0.3645	0.0693	0.0062
<b>Wang's price</b>	2.8779	1.9351	1.1123	0.4308	0.0948	0.0104
<b>Relative Difference</b>	-0.003%	0.25%	7.66%	18.18%	36.80%	68.51%

Figure 4: CEV model prices compared with Wang's probability distortion approach

<b>strike price</b>	18.000	19.000	20.000	21.000	22.000	23.000
<b>B-S price</b>	2.9132	2.1673	1.5413	1.0463	0.6782	0.4203
<b>Wang's price</b>	2.9267	2.1820	1.5629	1.0769	0.7068	0.4396
<b>Relative Difference</b>	0.46%	0.68%	1.41%	2.92%	4.22%	4.59%

Figure 5: Log-normal model results

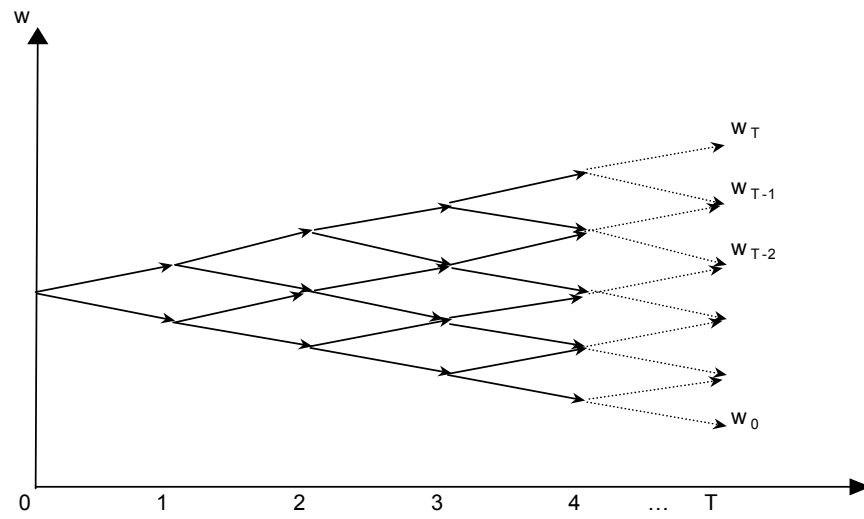


Figure 6: Recombining binomial lattice for  $T$  periods

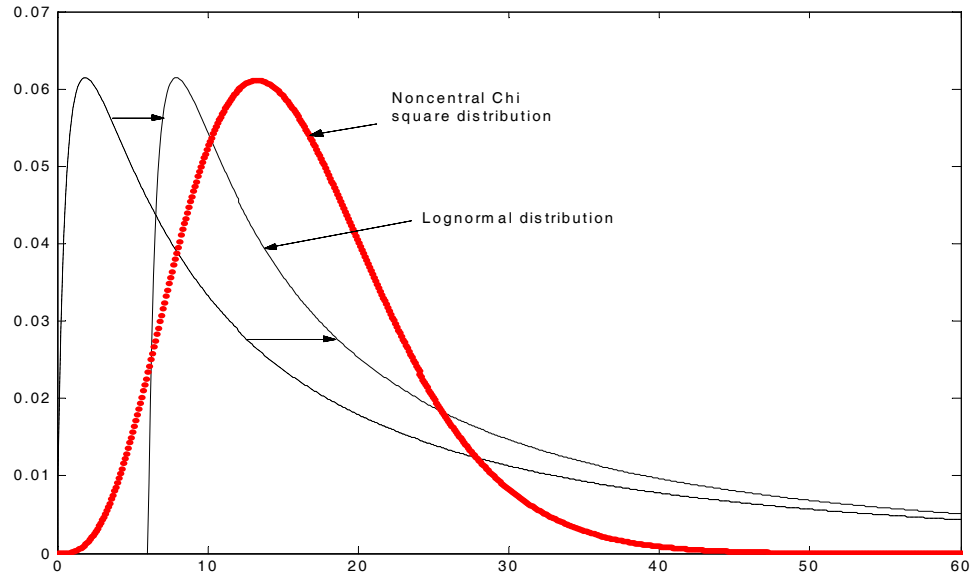


Figure 7: Shifting the mean of the lognormal distribution to match the mean of the noncentral Chi-square does not capture skewness and kurtosis